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# Quantum group $U_q(D_\ell)$ singular vectors in the Poincaré–Birkhoff–Witt basis

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**Abstract.** We give explicit expressions for the singular vectors of $U_q(D_\ell)$ in terms of the Poincaré–Birkhoff–Witt basis. We relate these expressions to those in terms of the simple root vectors.

## 1. Introduction

*Singular vectors* of Verma modules appeared in the representation theory of semisimple Lie algebras and groups, cf [1–4]. Since then singular vectors have also made a great impact in physics and not only from the point of view of applications. In fact, the generalization of singular vectors to other symmetry objects was made primarily in the (mathematical) physics literature starting with the paper [5], where singular vectors of the Virasoro algebra were crucially used. More explicit examples in this case were given in [6–10]. Further, singular vectors were given for Kac–Moody algebras [11, 12], the conformal superalgebra $su(2, 2/n)$ [13], the $N = 1$ super-Virasoro algebras [14, 15], quantum groups [16–18], $W$-algebras [19–21], the $N = 2$ super-Virasoro algebras [22–24], the $N = 4$ super-Virasoro algebras [25] and Kac–Moody superalgebras [26, 27]. Of course, the interest of physicists in the construction of singular (null) vectors arises mostly because of the numerous applications in, e.g., integrable theories [5, 6, 28–32], (super)conformal field theories [5, 33, 34], (super)string theories [35, 36], topological field theories [37] and Chern–Simons theory [38].

In this paper we consider singular vectors on Verma modules over the Drinfeld–Jimbo quantum groups [39,40]. These are $q$-deformations $U_q(\mathcal{G})$ of the universal enveloping algebras $U(\mathcal{G})$ of simple Lie algebras $\mathcal{G}$ and are called quantum groups [39], quantum universal enveloping algebras [41, 42] or just *quantum algebras*. This paper may be viewed as a natural continuation of the paper [17], where were given explicit formulae for the singular vectors of Verma modules over $U_q(\mathcal{G})$ for arbitrary $\mathcal{G}$ corresponding to a class of positive roots of $\mathcal{G}$, which were called straight roots, and some examples corresponding to arbitrary positive roots. Note that these results are complete only for $\mathcal{G} = A_\ell$ since in this case all positive roots are straight. The singular vectors were given only through the simple root vectors as

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in earlier work in the case $q = 1$, cf [43, 44]. (This basis turned out to be part of a more general basis introduced later in the context of quantum groups, though for other reasons, by Lusztig [45].) On the other hand, there were examples, in both the undeformed case [46] and the $q$-deformed case [47–49], when it was convenient to use singular vectors in the Poincaré–Birkhoff–Witt (PBW) basis. In particular, in [46, 47], the PBW basis was more suitable to solve the problem of explicitly reducing the representation spaces and directly obtaining the character formulae which give the spectrum of the unitary representations and thus are relevant for the applications to physics. In [48, 49], it turned out that the PBW basis was helpful in establishing a correspondence between elements in the universal enveloping algebra and the Gel'fand-(Weyl)-Zetlin basis vectors, which are used in many applications. In principle, the paper [18] generalizes the results of [11] (from which PBW singular vectors may be extracted) to the quantum group case; however, the formulae are not as explicit as is necessary for the applications.

Thus, the first result of this paper gives explicit expressions for the singular vectors of $U_q(D_\ell)$ in terms of the PBW basis. The second result relates these expressions to those in terms of the simple root vectors given. In fact, in this way we also obtain the expressions in terms of simple root vectors for the nonstraight roots which were not given in [17] (up to some special cases). The second result is also not known for $q = 1$.

## 2. Preliminaries

Let $\mathcal{G}$ be a complex simple Lie algebra with Chevalley generators $X_i^\pm$, $H_i$, $i = 1, \ldots, \ell =$ rank $\mathcal{G}$. Then the quantum algebra $U_q(\mathcal{G})$ is the $q$-deformation of the universal enveloping algebra $U(\mathcal{G})$ defined as the associative algebra over $\mathbb{C}$ with generators $X_i^\pm$, $K_i \equiv q_i^{H_i}$, $K_i^{-1} \equiv q_i^{-H_i}$ and with relations [40]

$$[K_i, K_j] = 0 \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1 \qquad K_i X_j^\pm K_i^{-1} = q_i^{\pm a_{ij}} X_j^\pm \tag{1a}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \tag{1b}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0 \qquad i \neq j \tag{1c}$$

where $q_i \equiv q^{(\alpha_i,\alpha_i)/2}$, $(a_{ij}) = (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathcal{G}$, $(\cdot, \cdot)$ is the scalar product of the roots normalized so that for the short roots $\alpha$ we have $(\alpha, \alpha) = 2$, $n = 1 - a_{ij}$,

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad [m]_q! = [m]_q[m-1]_q \ldots [1]_q \qquad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{1d}$$

Further we may omit the subscript $q$ from $[m]_q$ if no confusion could arise.

The above definition is valid also when $\mathcal{G}$ is an affine Kac–Moody algebra [39].

We use the standard decompositions into direct sums of vector subspaces $\mathcal{G} = \mathcal{H} \oplus \oplus_{\beta\in\Delta}\mathcal{G}_\beta = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$, $\mathcal{G}^\pm = \oplus_{\beta\in\Delta^\pm}\mathcal{G}_\beta$, where $\mathcal{H}$ is the Cartan subalgebra spanned by the elements $H_i$, $\Delta = \Delta^+ \cup \Delta^-$ is the root system of $\mathcal{G}$ and $\Delta^+$ and $\Delta^-$ are the sets of positive and negative roots, respectively; $\Delta_S$ will denote the set of simple roots of $\Delta$. We recall that $H_j$ correspond to the simple roots $\alpha_j$ of $\mathcal{G}$, and if $\beta^\vee = \sum_j n_j \alpha_j^\vee$, $\beta^\vee \equiv 2\beta/(\beta, \beta)$, then to $\beta$ corresponds $H_\beta = \sum_j n_j H_j$.

For the PBW basis of $U_q(\mathcal{G})$ besides $X_i^\pm$, $K_i^{\pm 1}$, we need also the Cartan–Weyl (CW) generators $X_\beta^\pm$ corresponding to the nonsimple roots $\beta \in \Delta^+$. Naturally, we shall use uniform notation, so that $X^\pm_{\alpha_i} \equiv X^\pm_i$. The CW generators $X^\pm_\beta$ are normalized so that [16, 40, 50]

$$[X^+_\beta, X^-_\beta] = \frac{K_\beta - K_\beta^{-1}}{q_\beta - q_\beta^{-1}} \qquad q_\beta \equiv q^{(\beta,\beta)/2}$$
$$K_\beta \equiv \prod_j K_j^{n_j(\beta,\beta)/(\alpha_j,\alpha_j)} (=q_\beta^{H_\beta}). \tag{2}$$

We shall not use the fact that the algebra $U_q(\mathcal{G})$ is a Hopf algebra and consequently we shall not introduce the corresponding structure.

The highest weight modules $V$ over $U_q(\mathcal{G})$ are given by their highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V$ such that

$$K_i v_0 = q_i^{\Lambda_i} v_0 \qquad X^+_i v_0 = 0 \qquad i = 1, \dots, \ell \qquad \Lambda_i \equiv (\Lambda, \alpha_i^\vee). \tag{3}$$

We start with the *Verma* modules $V^\Lambda$ such that $V^\Lambda \cong U_q(\mathcal{G}^-) \otimes v_0$. We recall several facts from [16]. The Verma module $V^\Lambda$ is reducible if there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$[(\Lambda + \rho, \beta^\vee) - m]_{q_\beta} = 0 \tag{4}$$

holds, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. If $q$ is not a root of unity then (4) is also a necessary condition for reducibility and then it may be rewritten as $2(\Lambda + \rho, \beta) = m(\beta, \beta)$. (In that case it is the generalization of the (necessary and sufficient) reducibility conditions for Verma modules over finite-dimensional $\mathcal{G}$ [1] and affine Lie algebras [51].) For uniformity we shall write the reducibility condition in the general form (4). If (4) holds then there exists a vector $v_s \in V^\Lambda$, called a *singular vector*, such that $v_s \notin \mathbb{C}v_0$, and

$$K_i v_s = q_i^{\Lambda_i - m(\beta,\alpha_i^\vee)} v_s \qquad i = 1, \dots, \ell \tag{5a}$$

$$X^+_i v_s = 0 \qquad i = 1, \dots, \ell. \tag{5b}$$

The space $U_q(\mathcal{G}^-)v_s$ is a proper submodule of $V^\Lambda$ isomorphic to the Verma module $V^{\Lambda - m\beta} = U_q(\mathcal{G}^-) \otimes v_0'$ where $v_0'$ is the highest-weight vector of $V^{\Lambda-m\beta}$; the isomorphism being realized by $v_s \mapsto 1 \otimes v_0'$. The singular vector is given by [16, 43, 44]:

$$v_s = v^{\beta,m} = \mathcal{P}^\beta_m \otimes v_0 \tag{6}$$

where $\mathcal{P}^\beta_m$ is a homogeneous polynomial of weight $m\beta$. The polynomial $\mathcal{P}^\beta_m$ is unique up to a nonzero multiplicative constant. The Verma module $V^\Lambda$ contains a unique proper maximal submodule $I^\Lambda$. Among the highest weight modules (HWMs) with highest weight $\Lambda$ there is a unique irreducible one, denoted by $L_\Lambda$, i.e. $L_\Lambda = V^\Lambda/I^\Lambda$. If $V^\Lambda$ is irreducible then $L_\Lambda = V^\Lambda$. Thus further we discuss $L_\Lambda$ for which $V^\Lambda$ is reducible. If $V^\Lambda$ is reducible w.r.t. to every simple root (and thus w.r.t. to all positive roots), then $L_\Lambda$ is a finite-dimensional HWM over $U_q(\mathcal{G})$ [52]. The representations of $U_q(\mathcal{G})$ are deformations of the representations of $U(\mathcal{G})$, and the latter are obtained from the former for $q \to 1$ [52].

In [17] the singular vectors were given only through the simple root vectors, namely

$$v^{\beta,m} = \mathcal{P}^\beta_m(X^-_1, \dots, X^-_\ell) \otimes v_0 \tag{7}$$

so $\mathcal{P}^\beta_m$ is a homogeneous polynomial in its variables of degrees $mn_i$, where $n_i \in \mathbb{Z}_+$ originate from $\beta = \sum n_i\alpha_i$.

Another restriction of [17] is that singular vectors were given only (up to a few special cases) for a class of positive roots of $\mathcal{G}$, which were called there straight roots. In order to introduce the latter class we first recall from [53] that every root may be expressed as the result of the action of an element of the Weyl group $W$ on some simple root. More explicitly, for any $\beta \in \Delta^+$ we have

$$\beta = w(\alpha_p) = s_{p_1} s_{p_2} \dots s_{p_r}(\alpha_p) \tag{8a}$$

$$s_\beta = w s_p w^{-1} = s_{p_1} \dots s_{p_r} s_p s_{p_r} \dots s_{p_1} \tag{8b}$$

where $\alpha_p$ is a simple root, the element $w \in W$ is written in a reduced form, i.e. in terms of the minimal possible number of the (generating $W$) simple reflections $s_i \equiv s_{\alpha_i}$, and the action of $s_\alpha$, $\alpha \in \Delta$, on $\mathcal{H}^*$ is given by $s_\alpha(\Lambda) = \Lambda - (\Lambda, \alpha^\vee)\alpha$. The positive root $\beta$ is called a *straight root* if all numbers $p, p_1, p_2, \dots, p_r$ in (8*a*) are different. Note that there may exist different forms of (8) involving other elements $w'$ and $\alpha_{p'}$; however, this definition does not depend on the choice of these elements. Obviously, any simple root is a straight root. Other easy examples of straight roots are those which are sums of simple roots with coefficients not exceeding unity, i.e. $\beta = \sum_i n_i \alpha_i$, with $n_i = 1$ or 0. All straight roots of the simply laced algebras $A_\ell$, $D_\ell$ and $E_\ell$ are of this form. In what follows we shall use also the following notation. A root $\gamma' \in \Delta^+$ is called a *subroot* of $\gamma'' \in \Delta^+$ if $\gamma'' - \gamma' \neq 0$ may be expressed as a linear combination of simple roots with non-negative coefficients.

In this paper we give explicit expressions for the singular vectors for all roots—for the straight roots in section 2 and for the nonstraight roots in section 3.

## 3. Singular vectors for the straight roots

### *3.1. Singular vectors in PBW basis*

In this paper we consider $U_q(\mathcal{G})$ when the deformation parameter $q$ is not a nontrivial root of unity. This generic case is very important for two reasons. First, for $q = 1$ all formulae are valid also for the undeformed case and formulae for the relation with [17] are new also for $q = 1$. Second, the formulae for the case when $q$ is a root of unity use the formulae for generic $q$ as important input as explained in [17].

Let $\mathcal{G} = D_\ell$, $\ell \geqslant 4$. Let $\alpha_i, i = 1, \dots, \ell$ be the simple roots, so that $(\alpha_i, \alpha_j) = -1$ if either $|i - j| = 1$, $i, j \neq \ell$ or $ij = \ell(\ell - 2)$ and $(\alpha_i, \alpha_j) = 2\delta_{ij}$ in other cases.

Then the positive roots are given as follows:

$$\begin{aligned}
&\alpha_{ij} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j && 1 \leqslant i < j \leqslant \ell - 2\\
&\beta_j = \alpha_j + \alpha_{j+1} + \dots + \alpha_{\ell-2} + \alpha_\ell && 1 \leqslant j \leqslant \ell - 2\\
&\tilde{\beta}_j = \alpha_j + \alpha_{j+1} + \dots + \alpha_{\ell-2} + \alpha_{\ell-1} && 1 \leqslant j \leqslant \ell - 2\\
&\beta_0 = \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell &&\\
&\gamma_j = \alpha_j + \alpha_{j+1} + \dots + \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell && 1 \leqslant j \leqslant \ell - 3\\
&\gamma_{ij} = \alpha_i + \alpha_{i+1} + \dots + 2(\alpha_j + \dots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell && 1 \leqslant i < j \leqslant \ell - 2.
\end{aligned} \tag{9}$$

We recall that the roots $\alpha_{ij}$, $\beta_j$, $\tilde{\beta}_j$ and $\beta_0$ are positive roots of various $A_n$ subalgebras. Thus, we have to consider only the roots $\gamma_j$ and $\gamma_{ij}$. We recall from [17] that $\gamma_j$ are straight, while $\gamma_{ij}$ are not straight.

In this section we deal with the straight roots $\gamma_j$. Now we recall that every root $\gamma_j$ is the highest straight root of a $D_{\ell-j+1}$ subalgebra of $D_\ell$. This means that it is enough to give the formula for the singular vector corresponding to the highest straight root $\gamma_1$.

Further we shall need the explicit expressions for the nonsimple-root CW generators of $U_q(\mathcal{G})$. Let $X^\pm_{i,j}$, $Y^\pm_j$, $\tilde{Y}^\pm_j$, $Y^\pm_0$, $Z^\pm_j$ and $Z^\pm_{i,j}$ be the CW generators corresponding respectively to the roots $\pm\alpha_{ij}$, $\pm\beta_j$, $\pm\tilde{\beta}_j$, $\pm\beta_0$, $\pm\gamma_j$ and $\pm\gamma_{ij}$. These generators are given recursively as follows (with $X^\pm_{jj} \equiv X^\pm_j$):

$$X^\pm_{ij} = \pm q^{\mp 1/2}(q^{1/2}X^\pm_i X^\pm_{i+1,j} - q^{-1/2}X^\pm_{i+1,j}X^\pm_i)$$
$$= \pm q^{\mp 1/2}(q^{1/2}X^\pm_{i,j-1}X^\pm_j - q^{-1/2}X^\pm_j X^\pm_{i,j-1}) \qquad 1 \leqslant i < j \leqslant \ell-2 \tag{10a}$$
$$Y^\pm_j = \pm q^{\mp 1/2}(q^{1/2}X^\pm_\ell X^\pm_{j,\ell-2} - q^{-1/2}X^\pm_{j,\ell-2}X^\pm_\ell)$$
$$= \pm q^{\mp 1/2}(q^{1/2}X^\pm_j Y^\pm_{j+1} - q^{-1/2}Y^\pm_{j+1}X^\pm_j) \qquad 1 \leqslant j \leqslant \ell-2 \tag{10b}$$
$$\tilde{Y}^\pm_j = \pm q^{\mp 1/2}(q^{1/2}X^\pm_{\ell-1} X^\pm_{j,\ell-2} - q^{-1/2}X^\pm_{j,\ell-2}X^\pm_{\ell-1})$$
$$= \pm q^{\mp 1/2}(q^{1/2}X^\pm_j \tilde{Y}^\pm_{j+1} - q^{-1/2}\tilde{Y}^\pm_{j+1}X^\pm_j) \qquad 1 \leqslant j \leqslant \ell-2 \tag{10c}$$
$$Y^\pm_0 = \pm q^{\mp 1/2}(q^{1/2}X^\pm_{\ell-1} Y^\pm_{\ell-2} - q^{-1/2}Y^\pm_{\ell-2}X^\pm_{\ell-1})$$
$$= \pm q^{\mp 1/2}(q^{1/2}X^\pm_\ell \tilde{Y}^\pm_{\ell-2} - q^{-1/2}\tilde{Y}^\pm_{\ell-2}X^\pm_\ell) \tag{10d}$$
$$Z^\pm_j = \pm q^{\mp 1/2}(q^{1/2}X^\pm_{j,\ell-3} Y^\pm_0 - q^{-1/2}Y^\pm_0 X^\pm_{j,\ell-3})$$
$$= \pm q^{\mp 1/2}(q^{1/2}X^\pm_\ell \tilde{Y}^\pm_j - q^{-1/2}\tilde{Y}^\pm_j X^\pm_\ell)$$
$$= \pm q^{\mp 1/2}(q^{1/2}X^\pm_{\ell-1} Y^\pm_j - q^{-1/2}\tilde{Y}^\pm_j X^\pm_{\ell-1}) \qquad 1 \leqslant j \leqslant \ell-3 \tag{10e}$$
$$Z^\pm_{ij} = \pm q^{\mp 1/2}(q^{1/2}Z^\pm_i X^\pm_{j,\ell-2} - q^{-1/2}X^\pm_{j,\ell-2}Z^\pm_i) \qquad 1 \leqslant i < j \leqslant \ell-2. \tag{10f}$$

Now the PBW basis of $U_q(\mathcal{G}^-)$ is given by the following monomials:

$$(X^-_{\ell-2})^{a_{\ell-2}}(X^-_{\ell-3,\ell-2})^{t_{\ell-3,\ell-2}}\ldots(X^-_{1,\ell-2})^{t_{1,\ell-2}}(\tilde{Y}^-_{\ell-2})^{\tilde{t}_{\ell-2}}(Y^-_{\ell-2})^{t_{\ell-2}}$$
$$\times(Z^-_{\ell-3,\ell-2})^{s_{\ell-3,\ell-2}}(Z^-_{\ell-4,\ell-2})^{s_{\ell-4,\ell-2}}$$
$$\times\ldots(Z^-_{1,\ell-2})^{s_{1,\ell-2}}(\tilde{Y}^-_{\ell-3})^{\tilde{t}_{\ell-3}}(Y^-_{\ell-3})^{t_{\ell-3}}$$
$$\times(Z^-_{\ell-4,\ell-3})^{s_{\ell-4,\ell-3}}\ldots(Z^-_{1,\ell-3})^{s_{1,\ell-3}}\ldots(\tilde{Y}^-_1)^{\tilde{t}_1}(Y^-_1)^{t_1}(Y^-_0)^t$$
$$\times(Z^-_{\ell-3})^{s_{\ell-3}}\ldots(Z^-_1)^{s_1}(X^-_\ell)^{a_\ell}(X^-_{\ell-1})^{a_{\ell-1}}(X^-_{\ell-3})^{a_{\ell-3}}$$
$$\times(X^-_{\ell-4,\ell-3})^{t_{\ell-4,\ell-3}}\ldots(X^-_{1,\ell-3})^{t_{1,\ell-3}}(X^-_{\ell-4})^{a_{\ell-4}}$$
$$\times\ldots(X^-_2)^{a_2}(X^-_{12})^{t_{12}}(X^-_1)^{a_1}. \tag{11}$$

These monomials are in the so-called normal order [50]. Namely, we put the simple root vectors $X^-_j$ in the order $X^-_{\ell-2}, X^-_\ell, X^-_{\ell-1}, X^-_{\ell-3}, \ldots, X^-_2, X^-_1$. Then we put a root vector $E^-_\alpha$ corresponding to the nonsimple root $\alpha$ between the root vectors $E^-_\beta$ and $E^-_\gamma$ if $\alpha = \beta + \gamma$, $\alpha, \beta, \gamma \in \Delta^+$. This order is not complete but this is not a problem, since when two roots are not ordered this means that the corresponding root vectors commute, e.g. $[X^-_i, X^-_{i-k,i+k}] = 0$, and $[Y^-_i, \tilde{Y}^-_i] = 0$, $1 \leqslant i \leqslant \ell-2$.

Let us have condition (4) fulfilled for $\gamma_1$, but not for any of its subroots $\gamma_i$, $i > 1$:

$$[(\Lambda+\rho, \gamma_1^\vee) - m]_q = 0 \qquad m \in \mathbb{N} \tag{12a}$$
$$[(\Lambda+\rho, \gamma_i^\vee) - m']_q \neq 0 \qquad \forall m' \in \mathbb{N}. \tag{12b}$$

(The necessity of the condition (12$b$) was explained in [17].) Let us denote the singular vector corresponding to (12$a$) by

$$v_s^{\gamma_1,m} = \sum_T D_T^{\gamma_1,m}(X^-_{\ell-2})^{a_{\ell-2}}(X^-_{\ell-3,\ell-2})^{t_{\ell-3,\ell-2}}\ldots(X^-_{1,\ell-2})^{t_{1,\ell-2}}(\tilde{Y}^-_{\ell-2})^{\tilde{t}_{\ell-2}}(Y^-_{\ell-2})^{t_{\ell-2}}$$
$$\times(Z^-_{\ell-3,\ell-2})^{s_{\ell-3,\ell-2}}(X^-_{\ell-4,\ell-2})^{s_{\ell-4,\ell-2}}\ldots(Z^-_{1,\ell-2})^{s_{1,\ell-2}}(\tilde{Y}^-_{\ell-3})^{\tilde{t}_{\ell-3}}(Y^-_{\ell-3})^{t_{\ell-3}}$$
$$\times(Z^-_{\ell-4,\ell-3})^{s_{\ell-4,\ell-3}}\ldots(Z^-_{1,\ell-3})^{s_{1,\ell-3}}\ldots(\tilde{Y}^-_1)^{\tilde{t}_1}(Y^-_1)^{t_1}(Y^-_0)^t$$
$$\times(Z^-_{\ell-3})^{s_{\ell-3}}\ldots(Z^-_1)^{s_1}(X^-_\ell)^{a_\ell}(X^-_{\ell-1})^{a_{\ell-1}}(X^-_{\ell-3})^{a_{\ell-3}}$$

$$\times(X^-_{\ell-4,\ell-3})^{t_{\ell-4,\ell-3}}\dots(X^-_{1,\ell-3})^{t_{1,\ell-3}}(X^-_{\ell-4})^{a_{\ell-4}}$$
$$\times\dots(X^-_2)^{a_2}(X^-_{12})^{t_{12}}(X^-_1)^{a_1}\otimes v_0 \tag{13}$$

where $T$ denotes the set of summation variables $a_i, t_{ij}, s_{ij}, \tilde{t}_i, t_i, s_i, t$, all of which are non-negative integers.

The derivation now proceeds as follows. We have to impose conditions (5) with $\beta \to \gamma_1$, $v_s \to v_s^{\gamma_1,m}$. (Inequalities (12*b*) mean that no other conditions need to be imposed.) First we impose conditions (5*a*). This restricts the linear combination to terms of weight $m\gamma_1$. In our parametrization these are the following $\ell$ conditions:

$$\begin{aligned} a_p &= m - \sum_{i=1}^{p}\left((t_i+\tilde{t}_i+s_i)+\sum_{j=p}^{\ell-2}t_{ij}+\sum_{j=p+1}^{\ell-2}s_{ij}+2\sum_{1\leqslant i<j\leqslant p}s_{ij}\right) \qquad 1\leqslant p\leqslant \ell-3\\ a_\ell &= m-\left(t+\sum_{i=1}^{\ell-2}t_i+\sum_{i=1}^{\ell-3}s_i+\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}\right)\\ a_{\ell-1} &= m-\left(t+\sum_{i=1}^{\ell-2}\tilde{t}_i+\sum_{i=1}^{\ell-3}s_i+\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}\right)\\ a_{\ell-2} &= m-\left(t+\sum_{i=1}^{\ell-3}(t_i+\tilde{t}_i)+\sum_{i=1}^{\ell-3}(s_i+t_{i,\ell-2})+2\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}\right). \end{aligned} \tag{14}$$

This eliminates the summation in $a_i$ in (13) and also restricts further the summation $t_{ij}, s_{ij}, \tilde{t}_i, t_i, s_i, t$ so that the $a_i$ in (14) would be all non-negative.

Next we impose conditions (5*b*). These $\ell$ conditions produce $\ell$ recursive relations, which are too cumbersome and we omit them. Solving these relations fixes the coefficients $D_T^{\gamma_1,m}$ completely and we obtain

$$\begin{aligned} D_T^{\gamma_1,m} &= D^\ell(-1)^{\sum_{i\leqslant j}s_{ij}}\frac{\prod_{p=2}^{\ell-3}\frac{[\tilde{a}_p]!}{[a_p]!}}{[t]!\prod_{j=2}^{\ell-2}[s_{1j}]![s_{j-1}]!\prod_{j=1}^{\ell-2}[t_j]![\tilde{t}_j]!\prod_{1\leqslant i<j\leqslant \ell-2}[t_{ij}]!}\\ &\quad\times\frac{q^{\boldsymbol{A}}q^{(\Lambda+\rho,a_\ell\alpha_\ell+a_{\ell-1}\alpha_{\ell-1})}}{[m-2t-\sum_{i=1}^{\ell-2}(t_i+\tilde{t}_i)-2\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}-2\sum_{i=1}^{\ell-3}s_i-\sum_{i=1}^{\ell-3}t_{i,\ell-2}]!}\\ &\quad\times\prod_{j=1}^{\ell-3}q^{a_j(\Lambda+\rho)(H^j)}\frac{\Gamma_q(\Lambda^j+j-a_j+t_{j-1,j})}{\Gamma_q(\Lambda^j+j+1)}\\ &\quad\times\frac{\Gamma_q(\Lambda_{\ell-1}+1-a_{\ell-1})\Gamma_q(\Lambda_\ell+1-a_\ell)}{\Gamma_q(\Lambda_{\ell-1}+2)\Gamma_q(\Lambda_\ell+2)} \end{aligned} \tag{15}$$

$$D^\ell\neq 0 \qquad \Lambda^r := (\Lambda,\beta^r) \qquad \text{with} \quad \beta^r := \alpha_1+\dots+\alpha_r$$

where

$$\tilde{a}_p = m-\sum_{i=1}^{p}\left((t_i+\tilde{t}_i+s_i)+\sum_{j=p+1}^{\ell-2}(t_{ij}+s_{ij})+2\sum_{1\leqslant i<j\leqslant p}s_{ij}\right) \qquad 1\leqslant p\leqslant \ell-3$$

and the factor $\boldsymbol{A}$ is given by

$$\begin{aligned} \boldsymbol{A} &= \sum_{1\leqslant i<j\leqslant \ell-2}\left\{t_{ij}\sum_{p=0}^{\ell-4}t^{p+j-1}+s_{ij}\sum_{p=0}^{\ell-4}s^{p+j-1}\right\}+\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}^2+\sum_{i=1}^{\ell-3}s_i^2\\ &\quad+\sum_{1\leqslant i<j\leqslant \ell-2}t_{ij}^2-\left((\ell-2)\sum_{1\leqslant i<j\leqslant \ell-2}t_{ij}+(\ell+1)\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}+\ell\sum_{i=1}^{\ell-3}s_i\right)m \end{aligned}$$

$$
\begin{aligned}
&+\sum_{1\leqslant i<j\leqslant \ell-2} t_{ij}\sum_{i=1}^{\ell-3}(t_i+d_i)+\sum_{i=1}^{\ell-4}(t_i+d_i)\sum_{j=1}^{\ell-3}(t_j+d_j)\\
&+\sum_{p=1}^{\ell-2}\left\{t_p\left(\sum_{j=1}^{p}t_j+\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}+\sum_{i=1}^{\ell-3}s_i-(\ell-p)m\right)\right.\\
&\left.+\tilde{t}_p\left(\sum_{j=1}^{p}\tilde{t}_j+\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}+\sum_{i=1}^{\ell-3}s_i-(\ell-p)m\right)\right\}\\
&+t\left(t+t_{\ell-2}+\tilde{t}_{\ell-2}+\sum_{i=1}^{\ell-3}s_i+\sum_{1\leqslant i<j\leqslant \ell-3}s_{ij}-3m\right)\\
&+\sum_{1\leqslant i<j\leqslant \ell-3}s_is_j+(\ell-2)\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}\sum_{k=1}^{\ell-3}s_k\\
&+\sum_{1\leqslant i<j\leqslant \ell-2}t_{ij}\left(\sum_{1\leqslant i<j\leqslant \ell-2}s_{ij}+\sum_{i=1}^{\ell-3}s_i\right)\\
&+\sum_{1\leqslant i<j\leqslant \ell-2}(j-i)t_{ij}+\sum_{\substack{1\leqslant i\leqslant \ell-4\\ i<j}}(\ell-j+3)s_{ij}+4s_{\ell-3,\ell-2}\\
&+\sum_{i=1}^{\ell-3}(\ell-i)s_i+\sum_{i=1}^{\ell-2}(\ell-i-1)(t_i+\tilde{t}_i)
\end{aligned}
\tag{16}
$$

where $t^b := \sum_{k=j+1}^{\ell-3} t_{bk}$.

Finally, we explain how to obtain the singular vectors for the roots $\gamma_i, i > 1$ from the above formulae. For this one has to replace $\ell \to \ell - i + 1$, and then to shift the enumeration of the roots, namely, to replace $1, \ldots, \ell - i + 1$ by $i, \ldots, \ell$.

### 3.2. Relation between the two expressions for the singular vectors

Here we would like to present the relation between the expressions for the singular vectors in the PBW basis given in (16) and in the simple root vector basis given in [17]. The latter formula is (cf formula (16) of [17])

$$
\begin{aligned}
v^{\gamma_1,m} = \sum_{k_1=0}^{m}\cdots\sum_{k_{\ell-1}=0}^{m} & d_{k_1,\ldots,k_{\ell-1}}(X_1^-)^{m-k_1}\ldots(X_{\ell-3}^-)^{m-k_{\ell-3}}(X_{\ell-1}^-)^{m-k_{\ell-1}}\\
&\times(X_\ell^-)^{m-k_{\ell-2}}(X_{\ell-2}^-)^{m}(X_\ell^-)^{k_{\ell-2}}(X_{\ell-1}^-)^{k_{\ell-1}}\\
&\times(X_{\ell-3}^-)^{k_{\ell-3}}\ldots(X_1^-)^{k_1}\otimes v_0
\end{aligned}
\tag{17a}
$$

$$
\begin{aligned}
d_{k_1\ldots k_{\ell-1}} &= d(-1)^{k_1+\cdots+k_{\ell-1}}\binom{m}{k_1}_q\cdots\binom{m}{k_{\ell-1}}_q\\
&\quad\times\frac{[(\Lambda+\rho,\beta^1)]_q}{[(\Lambda+\rho,\beta^1)-k_1]_q}\cdots\frac{[(\Lambda+\rho,\beta^{\ell-3})]_q}{[(\Lambda+\rho,\beta^{\ell-3})-k_{\ell-3}]_q}\\
&\quad\times\frac{[(\Lambda+\rho,\alpha_\ell)]_q}{[(\Lambda+\rho,\alpha_\ell)-k_{\ell-2}]_q}\frac{[(\Lambda+\rho,\alpha_{\ell-1})]_q}{[(\Lambda+\rho,\alpha_{\ell-1})-k_{\ell-1}]_q}\\
&= d(-1)^{k_1+\cdots+k_{\ell-1}}\binom{m}{k_1}_q\cdots\binom{m}{k_{\ell-1}}_q
\end{aligned}
$$

$$\times \frac{[\Lambda^1+1]_q}{[\Lambda^1-k_1]_q}\cdots\frac{[\Lambda^{\ell-3}+\ell-3]_q}{[\Lambda^{\ell-3}-k_{\ell-3}]_q}$$
$$\times \frac{[\Lambda_\ell+1]_q}{[\Lambda_\ell-k_{\ell-2}]_q}\frac{[\Lambda_{\ell-1}+1]_q}{[\Lambda_{\ell-1}-k_{\ell-1}]_q} \qquad d\neq 0 \tag{17b}$$

where $\Lambda_s = (\Lambda, \alpha_s)$.

The $D$-coefficients are given in term of the $d$-coefficients by the following formula:

$$D_T^{\gamma_1,m} = \frac{\prod_{p=2}^{\ell-3}\frac{[\tilde{a}_p]!}{[a_p]!}}{[t]!\prod_{j=2}^{\ell-2}[s_{1j}]![s_{j-1}]!\prod_{j=1}^{\ell-2}[t_j]![\tilde{t}_j]!\prod_{1\leqslant i<j\leqslant\ell-2}[t_{ij}]!}$$
$$\times\frac{(-1)^{\sum_{i=1}^{\ell}a_i}q^{\boldsymbol{A}}}{[m-2t-\sum_{i=1}^{\ell-2}(t_i+\tilde{t}_i)-2\sum_{1\leqslant i<j\leqslant\ell-2}s_{ij}-2\sum_{i=1}^{\ell-3}s_i-\sum_{i=1}^{\ell-3}t_{i,\ell-2}]!}$$
$$\times\sum_{k_1,k_2,\ldots,k_{\ell-1}} d_{k_1,k_2,\ldots,k_{\ell-1}}\prod_{p=1}^{\ell-3}\frac{[m-k_p]!q^{k_p(a_p-t_{p-1,p})}}{[a_p-t_{p-1,r}-k_p]!}$$
$$\times\frac{[m-k_{\ell-1}]!}{[a_{\ell-1}-k_{\ell-1}]!}\frac{[m-k_{\ell-2}]!}{[a_\ell-k_{\ell-2}]!}q^{(k_{\ell-1}a_{\ell-1}+k_{\ell-2}a_\ell)} \tag{18}$$

where $0\leqslant k_p\leqslant a_p$, $0\leqslant p\leqslant \ell-3$, $k_{\ell-1}\leqslant a_{\ell-1}$ and $k_{\ell-2}\leqslant a_\ell$.

To prove the above one can use the formula (following from (1$c$) and (10)):

$$\frac{V^m\,U^n}{[m]![n]!} = \sum_{0\leqslant p\leqslant\min(m,n)}(-1)^p q^{(m-p)(n-p)+p}\frac{U^{n-p}\,W^p\,V^{m-p}}{[n-p]![p]![m-p]!} \tag{19}$$

where the triples $U$, $V$, $W$ are given as follows: as $W$ runs over the vectors defined in (10), then $U$, $V$ run over the pairs which appear on the corresponding RHS, e.g. if $W = X_{ij}^-$ then either $(U,V) = (X_i^-, X_{i+1,j}^-)$ or $(U,V) = (X_{i,j-1}^-, X_j^-)$.

## 4. Singular vectors for the nonstraight roots

### *4.1. Singular vectors in the PBW basis*

The nonstraight roots of $D_\ell$ are given in (9). We shall also write them as

$$\gamma_{rp} = \sum_{j=r}^{\ell} n_j\alpha_j \qquad 1\leqslant r<p\leqslant\ell-2$$
$$n_j = \begin{cases}1 & \text{for}\quad r\leqslant j<p\\ 2 & \text{for}\quad p\leqslant j\leqslant\ell-2\\ 1 & \text{for}\quad j=\ell-1,\ell.\end{cases} \tag{20}$$

As in the case of straight roots we can use the fact that every root $\gamma_{rp}$ can be treated as the root $\gamma_{1p}$ of a $D_{\ell-r+1}$ subalgebra of $D_\ell$. This means that it would be enough to give the formula for the singular vector corresponding to the roots $\gamma_{1p}$. However, we shall not do this for these roots, since in any case it is not reduced to a single root.

Let us have condition (4) fulfilled for $\gamma_{rp}$, but not for any of its subroots. The singular vectors corresponding to these roots are given by

$$v_s^{\gamma_{rp},m} = \sum_T D_T^{\gamma_{rp},m}(X_{\ell-2}^-)^{2m-b_{\ell-2}}(X_{\ell-3,\ell-2}^-)^{t_{\ell-3,\ell-2}}\ldots(X_{r,\ell-2}^-)^{t_{r,\ell-2}}(\tilde{Y}_{\ell-2}^-)^{\tilde{t}_{\ell-2}}(Y_{\ell-2}^-)^{t_{\ell-2}}$$
$$\times(Z_{\ell-3,\ell-2}^-)^{s_{\ell-3,\ell-2}}(X_{\ell-4,\ell-2}^-)^{s_{\ell-4,\ell-2}}\ldots(Z_{r,\ell-2}^-)^{s_{r,\ell-2}}(\tilde{Y}_{\ell-3}^-)^{\tilde{t}_{\ell-3}}(Y_{\ell-3}^-)^{t_{\ell-3}}$$

$$\times(Z^-_{\ell-4,\ell-3})^{s_{\ell-4,\ell-3}}\dots(Z^-_{r,\ell-3})^{s_{r,\ell-3}}\dots(\tilde{Y}^-_r)^{\tilde{t}_r}(Y^-_r)^{t_r}(Y^-_0)^t$$
$$\times(Z^-_{\ell-3})^{s_{\ell-3}}\dots(Z^-_r)^{s_r}(X^-_\ell)^{m-b_\ell}(X^-_{\ell-1})^{m-b_{\ell-1}}(X^-_{\ell-3})^{mn_{\ell-3}-b_{\ell-3}}$$
$$\times(X^-_{\ell-4,\ell-3})^{t_{\ell-4,\ell-3}}\dots(X^-_{r,\ell-3})^{t_{r,\ell-3}}(X^-_{\ell-4})^{mn_{\ell-4}-b_{\ell-4}}$$
$$\times\dots(X^-_{r+1})^{mn_{r+1}-b_{r+1}}(X^-_{r,r+1})^{t_{r,r+1}}(X^-_r)^{m-b_r}\otimes v_0. \tag{21}$$

In (21) we have already imposed conditions (5$a$) and the summation is only over those elements of the PBW basis which have the weight $m\gamma_{rp}$. Further we impose (5$b$), the procedure being as in the case of the straight roots. Thus, the coefficients $D_T^{\gamma_{rp},m}$ are found to be

$$D_T^{\gamma_{rp},m} = D^{ns}(-1)^{\sum_{r\leqslant j} s_{ij}} \frac{\prod_{s=r+1}^{\ell-3}\frac{[mn_s-\tilde{b}_s]!}{[mn_s-b_s]!}}{[t]!\prod_{j=r+1}^{\ell-2}[s_{rj}]![s_{j-1}]!\prod_{j=r}^{\ell-2}[t_j]![\tilde{t}_j]!\prod_{r\leqslant i<j\leqslant\ell-2}[t_{ij}]!}$$
$$\times\frac{q^{A^{ns}}q^{(\Lambda+\rho,b_\ell\alpha_\ell+b_{\ell-1}\alpha_{\ell-1})}}{[2m-2t-\sum_{i=r}^{\ell-2}(t_i+\tilde{t}_i)-2\sum_{r\leqslant i<j\leqslant\ell-2}s_{ij}-2\sum_{i=r}^{\ell-3}s_i-\sum_{i=r}^{\ell-3}t_{i,\ell-2}]!}$$
$$\times\prod_{j=r}^{\ell-3}q^{(mn_j-b_j\Lambda'^j)}\frac{\Gamma_q(\Lambda'^j-mn_j+b_j+t_{j-1,j})}{\Gamma_q(\Lambda^j+1)}$$
$$\times\frac{\Gamma_q(\Lambda_{\ell-1}+1-m+b_{\ell-1})\Gamma_q(\Lambda_\ell+1-m+b_\ell)}{\Gamma_q(\Lambda_{\ell-1}+2)\Gamma_q(\Lambda_\ell+2)}$$
$$\Lambda'^j := \sum_{i=r}^{j} n_i(\Lambda_i+1) \qquad D^{ns}\neq 0 \tag{22}$$

where we have set for $r\leqslant p\leqslant\ell-3$

$$\tilde{b}_p = \sum_{i=r}^{p}\left((t_i+\tilde{t}_i+s_i)+\sum_{j=p+1}^{\ell-2}(t_{ij}+s_{ij})+2\sum_{r\leqslant i<j\leqslant p}s_{ij}\right)$$
$$b_p = \sum_{i=r}^{p}\left((t_i+\tilde{t}_i+s_i)+\sum_{j=p}^{\ell-2}t_{ij}+\sum_{j=p+1}^{\ell-2}s_{ij}+2\sum_{1\leqslant i<j\leqslant p}s_{ij}\right)$$
$$b_\ell = t+\sum_{i=r}^{\ell-2}t_i+\sum_{i=r}^{\ell-3}s_i+\sum_{r\leqslant i<j\leqslant\ell-2}s_{ij} \tag{23}$$
$$b_{\ell-1} = t+\sum_{i=r}^{\ell-2}\tilde{t}_i+\sum_{i=r}^{\ell-3}s_i+\sum_{r\leqslant i<j\leqslant\ell-2}s_{ij}$$
$$b_{\ell-2} = t+\sum_{i=r}^{\ell-3}(t_i+\tilde{t}_i)+\sum_{i=r}^{\ell-3}(s_i+t_{i,\ell-2})+2\sum_{r\leqslant i<j\leqslant\ell-2}s_{ij}.$$

### *4.2. Singular vectors in the simple root basis*

The singular vectors corresponding to the nonstraight roots, $\gamma_{rp}$, $1\leqslant r<p\leqslant\ell-2$, in the simple root basis are given by

$$v^{\gamma_{rp},m} = \sum_{k_r=0}^{m}\sum_{k_{r+1}=0}^{mn_{r+1}}\dots\sum_{k_{\ell-1}=0}^{m}d_{k_1,\dots,k_{\ell-1}}(X^-_r)^{m-k_r}(X^-_{r+1})^{mn_{r+1}-k_{r+1}}$$
$$\times\dots(X^-_{\ell-3})^{2m-k_{\ell-3}}(X^-_{\ell-1})^{m-k_{\ell-1}}(X^-_\ell)^{m-k_{\ell-2}}(X^-_{\ell-2})^{2m}(X^-_\ell)^{k_{\ell-2}}$$
$$\times(X^-_{\ell-1})^{k_{\ell-1}}(X^-_{\ell-3})^{k_{\ell-3}}\cdots(X^-_r)^{k_r}\otimes v_0. \tag{24}$$

The coefficients $d$ were not given in [17], but now using the PBW expression (21) for $v^{\gamma_{rp},m}$ we find that they are given by the following formula:

$$d_{k_r,\ldots,k_{\ell-1}} = (-1)^{k_r+\cdots+k_{\ell-1}} \times \sum_{\substack{mn_r-b_r\leqslant k_r\\ \vdots \\ mn_{\ell-3}-b_{\ell-3}\leqslant k_{\ell-3}}} \sum_{\substack{m-b_{\ell-1}\leqslant k_{\ell-1}\\ m-b_\ell\leqslant k_{\ell-2}}} D^{ns}$$
$$\times \prod_{j=r}^{\ell-3} \frac{q^{(mn_j-b_j)(1-k_j)-k_j}[mn_j-b_j]!}{[mn_j-k_j]![mn_j-\tilde{b}_j]![k_j-mn_j+b_j]!}$$
$$\times \frac{q^{(m-b_\ell)(1-k_{\ell-2})-k_{\ell-2}}}{[m-b_\ell]![k_{\ell-2}-m-b_\ell]![m-k_{\ell-2}]!}$$
$$\times \frac{q^{(m-b_{\ell-1})(1-k_{\ell-1})-k_{\ell-1}}}{[m-b_{\ell-1}]![k_{\ell-1}-m-b_{\ell-1}]![m-k_{\ell-1}]!}$$
$$\times \left[2m-2t-\sum_{i=r}^{\ell-2}(t_i+\tilde{t}_i)-2\sum_{r\leqslant i<j\leqslant\ell-2} s_{ij}-2\sum_{i=r}^{\ell-3} s_i-\sum_{i=r}^{\ell-3} t_{i,\ell-2}\right]!$$
$$\times [t]!\prod_{j=r+1}^{\ell-2}[s_{rj}]![s_{j-1}]!\prod_{j=r}^{\ell-2}[t_j]![\tilde{t}_j]!\prod_{r\leqslant i<j\leqslant\ell-2}[t_{ij}]!q^{-A^{ns}} \tag{25}$$

or more explicitly

$$d_{k_1,\ldots,k_{\ell-1}} = d^{ns}(-1)^{k_r+\cdots+k_{\ell-1}}\binom{mn_r}{k_r}_q\cdots\binom{mn_{\ell-1}}{k_{\ell-1}}_q$$
$$\times\frac{[(\Lambda+\rho,\beta^{r,r})]_q}{[(\Lambda+\rho,\beta^{r,r})-k_r]_q}\cdots\frac{[(\Lambda+\rho,\beta^{r,\ell-3})]_q}{[(\Lambda+\rho,\beta^{r,\ell-3})-k_{\ell-3}]_q}$$
$$\times\frac{[(\Lambda+\rho,\alpha_\ell)]_q}{[(\Lambda+\rho,\alpha_\ell)-k_{\ell-2}]_q}\frac{[(\Lambda+\rho,\alpha_{\ell-1})]_q}{[(\Lambda+\rho,\alpha_{\ell-1})-k_{\ell-1}]_q}$$
$$= d^{ns}(-1)^{k_r+\cdots+k_{\ell-1}}\binom{mn_r}{k_r}_q\cdots\binom{mn_{\ell-1}}{k_{\ell-1}}_q \tag{26}$$
$$\times\frac{[\Lambda'^r+n^r]_q}{[\Lambda'^r+n^r-k_1]_q}\cdots\frac{[\Lambda'^{\ell-3}+n^{\ell-3}]_q}{[\Lambda'^{\ell-3}+n^{\ell-3}-k_{\ell-3}]_q}$$
$$\times\frac{[\Lambda_\ell+1]_q}{[\Lambda_\ell+1-k_{\ell-2}]_q}\frac{[\Lambda_{\ell-1}+1]_q}{[\Lambda_{\ell-1}+1-k_{\ell-1}]_q}\qquad d^{ns}\neq 0$$

$$\beta^{r,j} := \sum_{i=r}^{j} n_i\alpha_i \qquad \Lambda'^j = (\Lambda,\beta^{r,j}), \qquad n^j := \sum_{i=r}^{j} n_i.$$

In the derivation of these formulae one can use (19).

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## References


[1] Bernstein I N, Gel'fand I M and Gel'fand S I 1971 *Funkts. Anal. Prilozh.* **5** 1–8 (Engl. Transl. 1971 *Funct. Anal. Appl.* **5** 1–8)
[2] Dixmier J 1974 *Algèbres Enveloppantes* (Paris: Gauthier-Villars)
[3] Kostant B 1975 *Lecture Notes in Mathematics* vol 466 (Berlin: Springer) p 101
[4] Zhelobenko D P 1976 *Math. USSR Izv.* **10** 1003
[5] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* **241** 333
[6] Furlan P, Sotkov G M and Todorov I T 1987 *J. Math. Phys.* **28** 1598
[7] Benoit L and Saint-Aubin Y 1988 *Phys. Lett.* B **215** 517
[8] Bauer M, Di Francesco P, Itzykson C and Zuber J B 1991 *Phys. Lett.* B **260** 323
Bauer M, Di Francesco P, Itzykson C and Zuber J B 1991 *Nucl. Phys.* **362** 515
[9] Kent A 1991 *Phys. Lett.* B **273** 56
Kent A 1992 *Phys. Lett.* B **278** 443
[10] Ganchev A Ch and Petkova V B 1992 *Phys. Lett.* B **293** 56
Ganchev A Ch and Petkova V B 1993 *Phys. Lett.* B **318** 77
[11] Malikov F G, Feigin B L and Fuchs D B 1986 *Funkts. Anal. Prilozh.* **20** 25 (Engl. Transl. 1986 *Funct. Anal. Appl.* **20** 103)
[12] Bauer M and Sochen N 1992 *Phys. Lett.* B **275** 82
Bauer M and Sochen N 1993 *Commun. Math. Phys.* **152** 127
[13] Dobrev V K and Petkova V B 1987 *Fortschr. Phys.* **35** 537
[14] Benoit L and Saint-Aubin Y 1991 *Lett. Math. Phys.* **23** 117
Benoit L and Saint-Aubin Y 1992 *Int. J. Mod. Phys.* A **7** 3023
Benoit L and Saint-Aubin Y 1994 *Int. J. Mod. Phys.* A **9** 547
[15] Watts G M T 1993 *Nucl. Phys.* **407** 213
[16] Dobrev V K 1991 *Proc. Int. Group Theory Conf. (St Andrews, 1989) (London Math. Soc. Lecture Note Series 159)* vol 1, ed C M Campbell *et al* (Cambridge: Cambridge University Press) p 87
[17] Dobrev V K 1991 *Lett. Math. Phys.* **22** 251
[18] Malikov F G 1992 *Int. J. Mod. Phys.* A **7** suppl 1B 623
[19] Bowcock P and Watts G M T 1992 *Phys. Lett.* B **297** 282
[20] Furlan P, Ganchev A Ch and Petkova V B 1993 *Phys. Lett.* B **318** 85
Furlan P, Ganchev A Ch and Petkova V B 1994 *Nucl. Phys.* **431** 622
[21] Bajnok Z 1994 *Phys. Lett.* B **320** 36
Bajnok Z 1994 *Phys. Lett.* B **329** 225
[22] Dörrzapf M 1996 *Commun. Math. Phys.* **180** 195
Dörrzapf M 1998 *Nucl. Phys.* B **529** 639
Dörrzapf M and Gato-Rivera B 1999 *Commun. Math. Phys.* **206** 493
(Dörrzapf M and Gato-Rivera B 1998 *Preprint* hep-th/9807234)
[23] Eholzer W and Gaberdiel M R 1997 *Commun. Math. Phys.* **186** 61
[24] Feigin B L, Semikhatov A M, Sirota V A and Tipunin I Yu 1998 *Nucl. Phys.* B **536** 617
Semikhatov A M 1997 *New Developments in Quantum Field Theory (Proc. NATO Workshop, Zakopane, 1997)* pp 329–39
(Semikhatov A M 1997 *Preprint* q-alg/9712024)
Semikhatov A M and Sirota V A 1997 *Preprint* hep-th/9712102
[25] Eguchi T and Taormina A 1987 *Phys. Lett.* B **196** 75
Eguchi T and Taormina A 1988 *Phys. Lett.* B **200** 315
Petersen J L and Taormina A 1990 *Nucl. Phys.* B **331** 556
Petersen J L and Taormina A 1990 *Nucl. Phys.* B **333** 833
[26] Bowcock P and Taormina A 1997 *Commun. Math. Phys.* **185** 467
Bowcock P, Hayes M and Taormina A 1998 *Nucl. Phys.* B **510** 739
[27] Ennes I P and Ramallo A V 1997 *Nucl. Phys.* B **502** 671
[28] Di Francesco P and Mathieu P 1992 *Phys. Lett.* B **278** 79
[29] Awata H, Matsuo Y, Odake S and Shiraishi J 1995 *Nucl. Phys.* B **449** 347
Awata H, Matsuo Y, Odake S and Shiraishi J 1995 *Preprint* hep-th/9503028
[30] Mathieu P and Watts G 1996 *Nucl. Phys.* B **475** 361
Mathieu P and Watts G 1998 *Nucl. Phys.* B **510** 577
[31] Babelon O, Bernard D and Smirnov F A 1997 *Commun. Math. Phys.* **186** 601
[32] Fioravanti D and Stanishkov M 1998 *Phys. Lett.* B **430** 109

[33] Huang C-S, Zhang D-H and Zheng Q-R 1992 *Phys. Rev.* D **46** 3503
Huang C-S, Zhang D-H and Zheng Q-R 1993 *Nucl. Phys.* B **389** 81
[34] Flohr M A I 1998 *Nucl. Phys.* B **514** 523
[35] Mukherji S, Mukhi S and Sen A 1991 *Phys. Lett.* B **266** 337
[36] Semikhatov A M 1996 *Preprint* hep-th/9610084
[37] Gato-Rivera B and Semikhatov A M 1993 *Nucl. Phys.* B **408** 133
Semikhatov A M 1994 *Mod. Phys. Lett.* A **9** 1867
[38] Kogan I I and Lewis A 1998 *Phys. Lett.* B **432** 77
[39] Drinfeld V G 1985 *Sov. Math.–Dokl.* **32** 2548
Drinfeld V G 1987 *Proc. Int. Congr. Mathematicians (Berkeley, CA, 1986)* vol 1 (Providence, RI: American Mathematical Society) p 798
[40] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[41] Reshetikhin N Yu 1987 Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links I & II *LOMI Leningrad Preprint* E-4-87
Reshetikhin N Yu 1987 Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links I & II *LOMI Leningrad Preprint* E-17-87
[42] Kirillov A N and Reshetikhin N Yu 1990 *New Developments in the Theory of Knots* ed T Kohno (Singapore: World Scientific) p 202
[43] Dobrev V K 1985 *J. Math. Phys.* **26** 235
[44] Dobrev V K 1988 *Rep. Math. Phys.* **25** 159
[45] Lusztig G 1990 *Prog. Theor. Phys. Suppl.* **102** 175
[46] Dobrev V K and Sezgin E 1991 *Int. J. Mod. Phys.* A **6** 4699
Dobrev V K and Sezgin E 1990 *Lecture Notes in Physics* vol 379 (Berlin: Springer) p 227
[47] Dobrev V K and Floreanini R 1994 *J. Phys. A: Math. Gen.* **27** 4831
[48] Dobrev V K and Truini P 1997 *J. Math. Phys.* **38** 3750
[49] Dobrev V K and Truini P 1997 *J. Math. Phys.* **38** 2631
[50] Tolstoy V N 1990 *Proc. Quantum Groups Workshop (Clausthal, 1989) (Lecture Notes in Physics vol 370)* ed H D Doebner and J D Hennig (Berlin: Springer) p 118
[51] Kac V G and Kazhdan D 1979 *Adv. Math.* **34** 97
[52] Rosso M 1987 *Commun. Math. Phys.* **117** 581
[53] Bourbaki N 1968 *Groupes at Algèbres de Lie* (Hermann: Paris) chs 4–6